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FAST TRACK COMMUNICATION

Extension of integrable equations

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Abstract
In this communication, we show that there is general construction to produce non-evolutionary integrable equations from a given integrable evolutionary equation. To support the main theorem, a few examples are explicitly given.

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1. Introduction

Recently, the authors of [1] applied Painlevé analysis to test the integrability of sixth-order nonlinear wave equations of the form

$$u_{xxxxxx} + au_x u_{xxxx} + bu_{xx}u_{xxx} + cu_x^2 u_{xx} + du_{tt} + eu_{xxxt} + fu_x u_{xt} + gu_t u_{xx} = 0, \tag{1}$$

where a, b, c, d, e, f and g are arbitrary parameters. They found four distinct cases to pass the Painlevé test. Three of these are previously known, and the fourth one is

$$\left(D_x^3 + 8u_x D_x + 4u_{xx}\right)\left(u_t + u_{xxx} + 6u_x^2\right) = 0, \tag{2}$$

which corresponds to the parameter $d = 0$ in (1).

When the parameter $d \neq 0$ in equation (1), using new notation a_i for its parameters we can rewrite it as

$$u_{tt} = a_1 u_{xxxxxx} + a_2 u_x u_{xxxx} + a_3 u_{xx}u_{xxx} + a_4 u_x^2 u_{xx} + a_5 u_{xxxt} + a_6 u_x u_{xt} + a_7 u_t u_{xx}.$$

Classification of such partial differential equations of second order (in time) that possess a hierarchy of infinitely many higher symmetries has been undertaken using the perturbative symmetry approach in [2, 3]. All homogeneous integrable equations of fourth and sixth order (in the space derivative) were listed out, and three new tenth-order integrable equations were found.

Equation (2) has generated enormous interest [4–8] since the communication [1] was posted on the arXiv. Kupershmidt [7] conjectured that nonholonomic perturbations for any bi-Hamiltonian system preserve integrability. Soon, the authors of [9] proved the conjecture.

In this communication, we show that there is a general construction to produce non-evolutionary integrable equations from a given integrable evolutionary equation. This construction does not require the integrable equations to be bi-Hamiltonian. We tackle the problem by answering the following question.

Given an integrable evolutionary differential equation $u_t + K = 0$, where K is a smooth function of x, u and x-derivatives of u up to an arbitrary but finite order, are there (pseudo)-differential operators $\mathcal{Q}$ such that the equation

$$\mathcal{Q}(u_t + K) = 0$$

is still integrable?

The answer is positive. In fact, there is more than one solution for operators $\mathcal{Q}$. In this communication, we prove that the factors of a Nijenhuis (see formula (12) for the definition) recursion operator of $u_t + K = 0$ are the possible operators $\mathcal{Q}$. The precise statement is as follows:

Theorem 1. *Let $\Re$ be a Nijenhuis recursion operator of the integrable evolutionary equation $u_t + K = 0$. Assume $\Re = \mathcal{P}\mathcal{Q}$, where $\mathcal{P}$ and $\mathcal{Q}$ are (pseudo)-differential operators (including the case when $\mathcal{P} = 1$). If $\mathcal{P}$ is non-degenerate, then*

$$\mathcal{Q}\mathcal{P}D_{\mathcal{Q}\Re^j(u_t+K)} - D_{\mathcal{Q}\Re^j(u_t+K)}\Re = 0, \qquad j = 0, 1, 2, \ldots, \tag{3}$$

for all solution of $\mathcal{Q}\Re^j(u_t + K) = 0$, where $D_\blacklozenge$ is the Fréchet derivative of $\blacklozenge$.

Here we should be careful about drawing a conclusion from (3) that $\Re$ is a recursion operator of the equation $\mathcal{Q}\Re^j(u_t + K) = 0$ for $j = 0, 1, 2, \ldots$, due to the nonlocal aspect of operators $\Re$ and $\mathcal{Q}\mathcal{P}$. More discussions on this can be found in the following section.

Consider the potential Korteweg–de Vries (PKdV) equation,

$$u_t + u_{xxx} + 6u_x^2 = 0. \tag{4}$$

It possesses a Nijenhuis recursion operator:

$$\Re_{\mathrm{PKdV}} = D_x^{-1}\left(D_x^3 + 8u_x D_x + 4u_{xx}\right),$$

where D_x^{-1} stands for the left inverse of D_x and $\mathcal{H}_{\mathrm{PKdV}} = D_x^{-1}$ is a Hamiltonian operator, and $\Im_{\mathrm{PKdV}} = D_x^3 + 8u_x D_x + 4u_{xx}$ is a symplectic operators. Equation (2) is obtained by $\Im_{\mathrm{PKdV}}$ acting on the PKdV equation. Since both $\Re_{\mathrm{PKdV}}$ and $\Im\mathcal{H}$ are weakly nonlocal, it follows from the above theorem that equation (2) is integrable and possesses the same recursion operator as the PKdV equation as noted in [1].

In the same spirit, the following equations are integrable:

$$D_x u_x^{-1} D_x \left(u_t + u_{xxx} - \tfrac{3}{2}u_x^{-1}u_{xx}^2\right) = 0,$$
$$(D_x \pm 2u)(u_t + u_{xxx} - 6u^2 u_x) = 0,$$
$$\left(D_x^2 + u_x\right)\left(u_t + u_{xxxxx} + 5u_x u_{xxx} + \tfrac{5}{3}u_x^3\right) = 0,$$
$$\left(D_x^5 + 5\left(u_x D_x^3 + D_x^3 u_x\right) - 3(u_{xxx}D_x + D_x u_{xxx}) + 8\left(u_x^2 D_x + D_x u_x^2\right)\right)(u_t + u_x) = 0.$$

We refer to section 3 for further details and more examples.

2. The proof of the theorem

In this section, we give the proof of theorem 1. First we introduce some basic concepts based on the book [10].

Consider $(1+1)$-dimensional differential equations with x, t being independent variables and u being a finite dimensional vector valued dependent variable

$$\Delta[u] = 0, \tag{5}$$

where $[u]$ means that the vector-valued smooth function Δ depends on u and its derivatives.

Definition 1. *An evolutionary vector field with characteristic $S[u]$ is a symmetry of system (5) if and only if*

$$D_\Delta[S] = 0, \tag{6}$$

where D_Δ is the Fréchet derivative of Δ.

The dual objects for symmetries are called cosymmetries. If $Q[u]$ is a cosymmetry of equation (5), it satisfies

$$D_\Delta^\star(Q) = 0, \tag{7}$$

where $D_\Delta^\star$ is the conjugate of the differential operator D_Δ.

Let E denote the Euler operator. For any cosymmetry $Q[u]$ of equation (5), we have

$$\mathrm{E}(Q \cdot \Delta) = D_Q^\star(\Delta) + D_\Delta^\star(Q) = 0.$$

Hence there exist $F[u]$ and $G[u]$ such that

$$Q \cdot \Delta = D_t F + D_x G. \tag{8}$$

Here F is called the conserved density and G is called the conserved flux. This is the characteristic form of a conservation law. Thus $Q[u]$ is also called the characteristic of a conservation law [10].

In this communication, system (5) is said to be integrable if it possesses infinitely many independent higher-order symmetries.

Often these symmetries can be generated by a recursion operator [11], which is a linear operator $\Re$ mapping a symmetry to a new symmetry. It satisfies

$$D_\Delta \Re = \tilde{\Re} D_\Delta \tag{9}$$

for all solutions of the equation, where $\tilde{\Re}$ is a linear operator. Suppose that

$$\Delta[u] = u_t - K(x, u, u_x, u_{xx}, \ldots, u_{xx\ldots x}) = 0 \tag{10}$$

is an evolutionary differential equation. Then $\tilde{\Re} = \Re$ and condition (9) reduces to

$$D_\Re[K] = [D_K, \Re]. \tag{11}$$

In some literature, formula (11) is taken as the definition of the recursion operator for evolution equation (10). However, we cannot take formula (9) as the definition for equation (5) unless we put constraints on the nonlocality of operator $\tilde{\Re}$. Otherwise, we can always find $\tilde{\Re}$ formally and this leads to a wrong conclusion: every operator is a recursion operator! In the case where $\Re$ is a weakly nonlocal differential operator [12], operator $\tilde{\Re}$ should also be weakly nonlocal. Here we refer to [13, 14] for another view of recursion operators, which are interpreted as Bäcklund autotransformations for the linearized equations.

All known recursion operators for nonlinear integrable equations are Nijenhuis operators, that is, for all evolutionary vector field with characteristic $S[u]$ operator $\Re$ satisfies

$$D_\Re[\Re S] - [D_{\Re S}, \Re] = \Re(D_\Re[S] - [D_S, \Re]). \tag{12}$$

Hence if the Nijenhuis operator $\Re$ is a recursion operator of $u_t = K$, then $\Re$ is a recursion operator for each of the evolution equations in the hierarchy $u_t = \Re^k K$, for $k = 0, 1, 2, \ldots$.

Interrelations among Hamiltonian and symplectic operators, and Nijenhuis operators were discovered by Gel'fand and Dorfman [15] and Fuchssteiner and Fokas [16, 17]. We refer to [18] for more details. Here we point out the relation between the recursion operator and Hamiltonian, and symplectic operators for equation (5).

We say that a Hamiltonian operator $\mathcal{H}$ mapping cosymmetries to symmetries of equation (5) is its Hamiltonian operator. It satisfies

$$D_\Delta \mathcal{H} = \tilde{\mathcal{H}} D_\Delta^\star \tag{13}$$

for all solutions of the equation, where $\tilde{\mathcal{H}}$ is a linear operator. Similarly, we can define a symplectic operator of equation (5), which satisfies

$$D_\Delta^\star \mathfrak{I} = \tilde{\mathfrak{I}} D_\Delta. \tag{14}$$

From the above formulae (13) and (14), we see that $\mathcal{H}\mathfrak{I}$ is a recursion operator satisfying formula (9) with $\tilde{\mathfrak{R}} = \tilde{\mathcal{H}}\tilde{\mathfrak{I}}$.

To prove theorem 1, it requires the following lemma.

Lemma 1. *Let $\mathfrak{R}$ be a Nijenhuis recursion operator of an integrable evolutionary equation $u_t + K = 0$. Then*

$$\mathfrak{R} D_{\mathfrak{R}^j(u_t+K)}[S] - D_{\mathfrak{R}^j(u_t+K)}[\mathfrak{R}S] + D_{\mathfrak{R}}[\mathfrak{R}^j(u_t+K)]S = 0, \qquad for \quad j = 0, 1, 2, \ldots. \tag{15}$$

Proof. Since $\mathfrak{R}$ is a recursion operator of $u_t + K = 0$, this implies that

$$D_{(u_t+K)}\mathfrak{R} - \mathfrak{R} D_{(u_t+K)} = D_{\mathfrak{R}}[u_t + K].$$

This is identity (15) for $j = 0$. Now we prove it for all $j > 0$ by induction. Assume formula (15) is valid for j and we compute

$$\begin{aligned} &\mathfrak{R} D_{\mathfrak{R}^{j+1}(u_t+K)}[S] - D_{\mathfrak{R}^{j+1}(u_t+K)}[\mathfrak{R}S] \\ &= \mathfrak{R}^2 D_{\mathfrak{R}^j(u_t+K)}[S] + \mathfrak{R} D_{\mathfrak{R}}[S]\mathfrak{R}^j(u_t+K) - \mathfrak{R} D_{\mathfrak{R}^j(u_t+K)}[\mathfrak{R}S] - D_{\mathfrak{R}}[\mathfrak{R}S]\mathfrak{R}^j(u_t+K) \\ &= \mathfrak{R}(\mathfrak{R} D_{\mathfrak{R}^j(u_t+K)}[S] - D_{\mathfrak{R}^j(u_t+K)}[\mathfrak{R}S]) + (\mathfrak{R} D_{\mathfrak{R}}[S] - D_{\mathfrak{R}}[\mathfrak{R}S])\mathfrak{R}^j(u_t+K). \end{aligned} \tag{16}$$

Since $\mathfrak{R}$ is a Nijenhuis operator, we know

$$\mathfrak{R} D_{\mathfrak{R}}[S] - D_{\mathfrak{R}}[\mathfrak{R}S] = \mathfrak{R}([D_S, \mathfrak{R}]) - [D_{\mathfrak{R}S}, \mathfrak{R}] = \mathfrak{R} D_S \mathfrak{R} - \mathfrak{R}^2 D_S - D_{\mathfrak{R}S}\mathfrak{R} + \mathfrak{R} D_{\mathfrak{R}S}.$$

This leads to

$$(\mathfrak{R} D_{\mathfrak{R}}[S] - D_{\mathfrak{R}}[\mathfrak{R}S])\mathfrak{R}^j(u_t+K) = \mathfrak{R} D_{\mathfrak{R}}[\mathfrak{R}^j(u_t+K)]S - D_{\mathfrak{R}}[\mathfrak{R}^{j+1}(u_t+K)](S).$$

Therefore, (16) becomes

$$\mathfrak{R}(\mathfrak{R} D_{\mathfrak{R}^j(u_t+K)}[S] - D_{\mathfrak{R}^j(u_t+K)}[\mathfrak{R}S] + D_{\mathfrak{R}}[\mathfrak{R}^j(u_t+K)]S) - D_{\mathfrak{R}}[\mathfrak{R}^{j+1}(u_t+K)](S),$$

which is equivalent to

$$\begin{aligned} &\mathfrak{R} D_{\mathfrak{R}^{j+1}(u_t+K)}[S] - D_{\mathfrak{R}^{j+1}(u_t+K)}[\mathfrak{R}S] + D_{\mathfrak{R}}[\mathfrak{R}^{j+1}(u_t+K)](S) \\ &= \mathfrak{R}(\mathfrak{R} D_{\mathfrak{R}^j(u_t+K)}[S] - D_{\mathfrak{R}^j(u_t+K)}[\mathfrak{R}S] + D_{\mathfrak{R}}[\mathfrak{R}^j(u_t+K)]S). \end{aligned}$$

Using the induction assumption, we proved the identity. □

Proof of theorem 1. Note that for all solutions of $\mathcal{Q}\mathfrak{R}^j(u_t + K) = 0$ and any evolutionary vector field with characteristic S we have

$$D_{\mathfrak{R}^{j+1}(u_t+K)}[S] = D_{\mathcal{P}}[S](\mathcal{Q}\mathfrak{R}^j(u_t+K)) + \mathcal{P} D_{\mathcal{Q}\mathfrak{R}^j(u_t+K)}[S] = \mathcal{P} D_{\mathcal{Q}\mathfrak{R}^j(u_t+K)}[S].$$

Thus using lemma 1, we obtain

$$\begin{aligned} &\mathcal{P}(\mathcal{Q}\mathcal{P} D_{\mathcal{Q}\mathfrak{R}^j(u_t+K)}[S] - D_{\mathcal{Q}\mathfrak{R}^j(u_t+K)}[\mathfrak{R}S]) \\ &= \mathfrak{R}\mathcal{P} D_{\mathcal{Q}\mathfrak{R}^j(u_t+K)}[S] - \mathcal{P} D_{\mathcal{Q}\mathfrak{R}^j(u_t+K)}[\mathfrak{R}S] \\ &= \mathfrak{R} D_{\mathfrak{R}^{j+1}(u_t+K)}[S] - D_{\mathfrak{R}^{j+1}(u_t+K)}[\mathfrak{R}S] = -D_{\mathfrak{R}}[\mathfrak{R}^{j+1}(u_t+K)](S) = 0. \end{aligned}$$

Due to non-degeneracy of operator $\mathcal{P}$, we obtain formula (3) and hence the statement is proved. □

The majority of known recursion operators $\Re$ for evolutionary equations can be written as the products of weakly nonlocal operators [19]. In this case, we can draw the following conclusion from formula (3):

Corollary 1. *Under the conditions of theorem 1, if both $\mathcal{P}$ and $\mathcal{Q}$ are weakly nonlocal, then $\Re$ is a recursion operator of the equation $\mathcal{Q}\Re^j(u_t + K) = 0$ for $j = 0, 1, 2, \ldots$.*

Now we compute the conjugation of formula (3) and we have

$$\Re^\star D^\star_{\mathcal{Q}\Re^j(u_t+K)} - D^\star_{\mathcal{Q}\Re^j(u_t+K)}\mathcal{P}^\star\mathcal{Q}^\star = 0. \tag{17}$$

Note that if Q is a cosymmetry of $\mathcal{Q}\Re^j(u_t + K) = 0$, so is $\mathcal{P}^\star\mathcal{Q}^\star Q$ if it is local. So operator $\mathcal{P}^\star\mathcal{Q}^\star$ mapping its cosymmetry to a new cosymmetry.

The cosymmetries of the extended integrable equation $\mathcal{Q}\Re^j(u_t + K) = 0$ are closely related to those of the equation $u_t + K = 0$.

Corollary 2. *Under the conditions of theorem 1, if Q is a cosymmetry of $\mathcal{Q}\Re^j(u_t + K) = 0$, then $\Re^{j\star}\mathcal{Q}^\star Q$ is a cosymmetry of $u_t + K = 0$. If $\Re^{(j+1)\star}\bar{Q}$ is a cosymmetry of $u_t + K = 0$, then $\mathcal{P}^\star\bar{Q}$ is a cosymmetry of $\mathcal{Q}\Re^j(u_t + K) = 0$.*

Proof. Let E be the Euler operator and Q be a cosymmetry of $\mathcal{Q}\Re^j(u_t + K) = 0$. We have

$$0 = \mathrm{E}(Q \cdot \mathcal{Q}\Re^j(u_t + K)) = \mathrm{E}(\Re^{j\star}\mathcal{Q}^\star Q \cdot (u_t + K)),$$

that is, $\Re^{j\star}\mathcal{Q}^\star Q$ is a cosymmetry of $u_t + K = 0$. On the other hand, let $\Re^{(j+1)\star}\bar{Q}$ be a cosymmetry for the equation $u_t + K = 0$, that is,

$$0 = \mathrm{E}(\Re^{(j+1)\star}\bar{Q} \cdot (u_t + K)) = \mathrm{E}(\Re^{j\star}\mathcal{Q}^\star\mathcal{P}^\star\bar{Q} \cdot (u_t + K)) = \mathrm{E}(\mathcal{P}^\star\bar{Q} \cdot \mathcal{Q}\Re^j(u_t + K)).$$

This implies that $\mathcal{P}^\star\bar{Q}$ is a cosymmetry of $\mathcal{Q}\Re^j(u_t + K) = 0$. □

In our first example, we know u_{xx} is a cosymmetry of the PKdV equation (4). Then $-u_x$ is a cosymmetry of equation (2). Indeed,

$$-u_x \cdot \left(D_x^3 + 8u_x D_x + 4u_{xx}\right)\left(u_t + u_{xxx} + 6u_x^2\right)$$
$$= D_t\left(\tfrac{1}{2}u_{xx}^2 - 2u_x^3\right) + D_x\left(-u_x u_{xxt} - 2u_x^2 u_t - u_x u_{5x} + u_{xx}u_{4x} - \tfrac{1}{2}u_{3x}^2 - 20u_x^2 u_{3x} - 30u_x^4\right).$$

Equation (2) can be written in the conserved form as

$$\left(D_x^3 + 8u_x D_x + 4u_{xx}\right)\left(u_t + u_{xxx} + 6u_x^2\right)$$
$$= D_t\left(2u_x^2\right) + D_x\left(D_x^2\left(u_t + u_{xxx} + 6u_x^2\right) + 4u_x u_t + 8u_x u_{xxx} - 2u_{xx}^2 + 40u_x^3\right).$$

This implies 1 is a cosymmetry of equation (2), which corresponds to a cosymmetry $-4u_{xx}$ for the PKdV equation (4).

3. More examples

In this section, we list a few scalar integrable equations obtained using theorem 1 when $\Re = \mathcal{P}\mathcal{Q}$ is weakly nonlocal and operator $\mathcal{Q}$ is local. It follows from corollary 1 that operator $\Re$ is a recursion operator of the equations obtained.

3.1. The potential Burgers' equation

Consider the potential Burgers' equation $u_t + u_{xx} + u_x^2 = 0$ with a Nijenhuis recursion operator

$$\Re = D_x + u_x.$$

Its hierarchy of commuting local symmetries is give by $\Re^j u_x$ for $j = 0, 1, 2, \ldots$. The flows given by any linear combinations of such symmetries, i.e., $u_t = K = \sum_{i=1}^n \Re^{j_i} u_x$ are integrable and share the same recursion operator $\Re$. It follows from theorem 1 that

$$(D_x + u_x)^l (u_t + K) = 0, \qquad K = \sum_{i=1}^n \lambda_i \Re^{j_i} u_x, \tag{18}$$

for all $l = 0, 1, 2, \ldots$, and constants λ_i are integrable sharing the same recursion operator $\Re$. Due to formula (3) and the linearity of Fréchet derivative, any linear combinations of extended equations (18) are also integrable. As pointed out in [20], there is a two-fold integrable hierarchy for equation (2). Here the freedom of both l and j reflects the same fact for the extended integrable potential Burgers' equation.

Indeed, by the Cole–Hopf transformation $w = \exp(u)$, we can linearize equation (18). In particular, the equation $(D_x + u_x)(u_t + u_x) = 0$ can be transformed into $w_{xt} + w_{xx} = 0$.

3.2. The Schwarzian Korteweg–de Vries equation

The Schwarzian Korteweg–de Vries equation, $u_t + u_{xxx} - \frac{3}{2} u_x^{-1} u_{xx}^2 = 0$, possesses a Nijenhuis recursion operator $\Re = \mathcal{H}\Im$, where $\mathcal{H} = u_x D_x^{-1} u_x$ and

$$\Im = u_x^{-2} D_x^3 - 3u_x^{-3} u_{xx} D_x^2 + \left(3u_x^{-4} u_{xx}^2 - u_x^{-3} u_{xxx}\right) D_x = D_x u_x^{-1} D_x u_x^{-1} D_x.$$

We can obtain infinitely many integrable non-evolutionary equations as in the previous section via a local operator $\Im$. A typical example is

$$\Im\left(u_t + u_{xxx} - \tfrac{3}{2} u_x^{-1} u_{xx}^2\right) = 0.$$

Since operator $\Im$ can be factorized into local operators, we can also obtain integrable non-evolutionary equations via its factors. Therefore, the equation

$$D_x u_x^{-1} D_x \left(u_t + u_{xxx} - \tfrac{3}{2} u_x^{-1} u_{xx}^2\right) = 0$$

is also integrable.

3.3. The modified Korteweg–de Vries equation

The modified Korteweg–de Vries equation, $u_t + u_{xxx} - 6u^2 u_x = 0$, possesses a Nijenhuis recursion operator:

$$\begin{aligned}\Re &= D_x^2 - 4u^2 - 4u_x D_x^{-1} u = D_x\left(D_x - 4u D_x^{-1} u\right) \\ &= D_x (D_x - 2u) D_x^{-1} (D_x + 2u) = D_x (D_x + 2u) D_x^{-1} (D_x - 2u).\end{aligned}$$

Similar to sections 3.1 and 3.2, we can extend it to non-evolutionary integrable equations. One example is

$$(D_x \pm 2u)(u_t + u_{xxx} - 6u^2 u_x) = 0.$$

3.4. The potential Sawada–Kotera equation

The potential Sawada–Kotera equation, $u_t + u_{xxxxx} + 5u_x u_{xxx} + \frac{5}{3}u_x^3 = 0$, possesses a Nijenhuis recursion operator $\mathfrak{R} = \mathcal{H}\mathfrak{I}$, where

$$\mathcal{H} = D_x + 2u_x D_x^{-1} + 2D_x^{-1}u_x \qquad \text{and} \qquad \mathfrak{I} = \left(D_x^2 + u_x\right)D_x\left(D_x^2 + u_x\right).$$

Typical integrable examples are

$$\mathfrak{I}\left(u_t + u_{xxxxx} + 5u_x u_{xxx} + \tfrac{5}{3}u_x^3\right) = 0, \tag{19}$$

$$\left(D_x^2 + u_x\right)\left(u_t + u_{xxxxx} + 5u_x u_{xxx} + \tfrac{5}{3}u_x^3\right) = 0. \tag{20}$$

3.5. The potential Kaup–Kupershmidt equation

The potential Kaup–Kupershmidt equation,

$$u_t + u_{xxxxx} + 10u_x u_{xxx} + \tfrac{15}{2}u_{xx}^2 + \tfrac{20}{3}u_x^3 = 0,$$

possesses a Nijenhuis recursion operator $\mathfrak{R} = \mathcal{H}\mathfrak{I}$, where $\mathcal{H} = D_x + u_x D_x^{-1} + D_x^{-1}u_x$ and

$$\mathfrak{I} = D_x^5 + 5\left(u_x D_x^3 + D_x^3 u_x\right) - 3(u_{xxx}D_x + D_x u_{xxx}) + 8\left(u_x^2 D_x + D_x u_x^2\right).$$

According to theorem 1, we can extend infinitely many integrable non-evolutionary equations. Typical examples are

$$\mathfrak{I}(u_t + u_x) = 0, \tag{21}$$

$$\mathfrak{I}\left(u_t + u_{xxxxx} + 5u_x u_{xxx} + \tfrac{5}{3}u_x^3\right) = 0. \tag{22}$$

Equation (21) is equivalent to system (2.14) in [21] under simple scaling transformation. Thus it has a reciprocal link to the Degasperis–Procesi equation:

$$u_t - u_{xxt} + 4uu_x = 3u_x u_{xx} + uu_{xxx}.$$

4. Conclusion and further research

In this communication, we give a general construction to extend from integrable evolutionary equations to integrable non-evolutionary equations. There are potential applications of such a construction. In general, it is much harder to study the algebraic and geometric structures of integrable non-evolutionary equations. For a given such equation, it is advisable to transform it into the form $\mathcal{Q}(u_t + K) = 0$. More research will be undertaken in this direction.

For non-evolutionary equations, it is convenient to have formula (9) to determine whether a given operator is a recursion operator or not. Due to the nonlocality of recursion operators $\mathfrak{R}$, we need to check whether it maps a symmetry to a new symmetry. This is equivalent to adding the constraints on operator $\tilde{\mathfrak{R}}$. In this communication, we applied theorem 1 for recursion operators being the products of weakly nonlocal operators. For a given evolutionary equation, this theorem can be applied for a given recursion operator of other types.

Under the construction, the extended non-evolutionary equations share the same recursion operator. But their solutions are different since the kernels of the differential operators involved are different; see equations (19) and (20). It would be interesting to see how the properties such as Lax representations and solutions of such equations are related.

Due to the nonlocality, the factorization of Nijenhuis operators is an unsolved problem. For a given operator, we can try to represent it as a product of lower-order operators in the same spirit for linear differential operators [22]. For the Nijenhuis operator $\mathfrak{R} = \mathcal{P}\mathcal{Q}$, we do not require Hamiltonian and symplectic properties for operators $\mathcal{P}$ and $\mathcal{Q}$. In fact, operator $\mathcal{P}$ has the property that $[\mathcal{P}F, \mathcal{P}G] \in \mathrm{Im}\mathcal{P}$ for any $F[u]$ and $G[u]$. This can be viewed as the generalization of Hamiltonian operators [23–25]. We are going to study the properties of operators $\mathcal{P}$ and $\mathcal{Q}$, and their interrelation with Nijenhuis operators, which is helpful for the factorization of the Nijenhuis operators. Some results have been obtained in this direction and will be published in the future.

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